Complex Numbers (Year 1)

The definition of i

Definition of i

$i = \sqrt{-1}$

$i^2 = -1$

We are often asked to do calculate powers of i. Use the fact that:

- $i^2 = -1$

Step 1: Use the indices rules $x^{ab} = (x^a)^b$ Where a is a power of 2

Step 2: Simplify what is inside the bracket using the fact that $i^2 = -1$

Alternative method: Use powers of 4 instead of 2 but then you need to also know that $i^3 = -i$ and $i^4 = 1$

Examples

Example 1	Example 2	Example 3
Simplify: i^{52}	Simplify: i^{31}	Simplify: $\sqrt{-10}\sqrt{-40}$
$= (i^2)^{26}$ $= (-1)^{26}$ A negative number to an even power is positive $= \mathbf{1}$	$= i^{30}i$ $= (i^2)^{15}i$ $= (-1)^{15}i$ $= -i$	$\sqrt{-10}\sqrt{-40}$ $= \sqrt{10}\sqrt{-1} \times \sqrt{40}\sqrt{-1}$ $= \sqrt{10}i \times \sqrt{40}i$ $= \sqrt{400}i^2$ $= (20)(-1) = \mathbf{-20}$ Do not make the mistake $\sqrt{-10}\sqrt{-40} = \sqrt{400} = 20$

Cartesian Form

We usually use the letter z to denote a complex number

z = real part $\pm$ i (imaginary part)

Note: This can also be written as z = real part $\pm$ (imaginary part) i

We usually use the letters a and b or x and y

$z = a + ib$ or $z = x + iy$

Addition/Subtraction

When we add/subtract we **combine** the real parts and then the imaginary parts

Adding:

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

Subtracting:

$$(a + bi) - (c + di) = (a - c) + (b - d)i$$

Examples

Example 1: Addition	Example 2: Subtraction	
Simplify: $(3 - 2i) + (4 + 3i)$	Simplify $(3 - 2i) - (4 + 3i)$	$z_1 = 2 + 3i, z_2 = -3 + 5i$ Find $z_1 - z_2$
$(3 - 2i) + (4 + 3i)$ $= (3 + 4) + (-2 + 3)i$ $= \mathbf{7 + i}$	$(3 - 2i) - (4 + 3i)$ $= (3 - 4) + (-2 - 3)i$ $= \mathbf{-1 - 5i}$	$(2 + 3i) - (-3 + 5i)$ $= (2 - -3) + (3 - 5)i$ $= \mathbf{5 - 2i}$

Multiplication/Division

Multiplying:

$$(a + bi)(c + di) = ac + adi + cbi - bd$$

Dividing:

$$\frac{a + ib}{c + id}$$

Multiply by the complex conjugate of the denominator

$$\frac{a + ib}{c + id} \times \frac{c - id}{c - id} = \frac{(a + ib)(c - id)}{c^2 + d^2}$$

Examples

Example 1: Multiplication	Example 2: Division
Simplify: $(3 - 2i)(4 + 3i)$	Simplify: $\frac{3 - 2i}{4 + 3i}$
$(3 - 2i)(4 + 3i)$ $12 + 9i - 8i - 6i^2$ $= 12 + 9i - 8i - 6(-1)$ $= \mathbf{18 + i}$	$\frac{3 - 2i}{4 + 3i}$ $= \frac{3 - 2i}{4 + 3i} \times \frac{4 - 3i}{4 - 3i}$ $= \frac{12 - 9i - 8i - 6}{16 + 9} = \frac{6 - 17i}{25}$ $= \frac{6}{25} - \frac{17}{25}i$

Complex Conjugate

Complex conjugate means we swap the sign of the imaginary part

$a + bi \Rightarrow a - bi$

$a - bi \Rightarrow a + bi$

Notation: z^* or $\bar{z}$

Conjugate property shortcuts to make your calculations quicker:

- $(z \pm w)^* = z^* \pm w^*$
- $(zw)^* = z^* w^*$ (find the conjugate of each and multiply them together)
- $\left(\frac{z}{w}\right)^* = \frac{z^*}{w^*}$ if $w \neq 0$ (find the conjugate of each and divide them)
- $z + z^* = 2Re(z)$ and $z - z^* = 2Im(z)$
- $z.z^* = |z|^2$

Examples

Example 1	Example 2	Example 3
Find the complex conjugate and modulus of $z = 2 - 3i$	$z = 3 - 8i$ Find $z + z*$	Given: $z = 2 + 3i$, $w = 5 - 8i$ Find $(z + w)^*$
Complex conjugate: $\mathbf{2 + 3i}$ Mod $= \sqrt{2^2 + (-3)^2} = \sqrt{\mathbf{13}}$	$(3 - 8i) + (3 + 8i)$ $= \mathbf{6}$	$(z + w)$ $= (2 + 3i) + (5 - 8i)$ $= 7 - 5i$ $(z + w)^* = \mathbf{7 + 5i}$

Equating Real and Imaginary Coefficients (Comparing Coefficients)

This method is used to find unknowns in equations in order to solve equations or to find square roots

Example 1 :	Example 2:	Example 3: Finding Square Roots
Let $z = x + iy$. Find the values of x and y if $(1 - i)z = 1 - 3i$	Given that $\frac{z}{z-8} = -1 - 2i$. Find z in the form $a + ib$	Find the square roots of $8 - 6i$
Let $z = x + iy$ $(1 - i)(x + iy) = 1 - 3i$ $x + iy - ix + y = 1 - 3i$ $(x + y) + i(-x + y) = 1 - 3i$ Equating real and imaginary parts gives: $x + y = 1$ and $-x + y = -3$ Solving simultaneously gives: $\mathbf{x = 2}$ **and** $\mathbf{y = -1}$	$z = (-1 - 2i)(z - 8)$ Let $z = a + ib$ $a + ib = (-1 - 2i)(a + ib - 8)$ $a + ib = -a - ib + 8 - 2ai + 2b + 16i$ $a + ib = (-a + 2b + 8) + i(-b - 2a + 16)$ Equating real and imaginary parts gives: $-a + 2b + 8 = a$ and $-b - 2a + 16 = b$ Simplifying the above equations further: $-2a + 2b = -8$ and $-2a - 2b = -16$ Solving simultaneously gives: $a = 6, b = 2$ $\Rightarrow \mathbf{z = 6 + 2i}$	$z = \sqrt{8 - 6i}$ $z^2 = 8 - 6i$ Let $z = x + iy$ $(x + iy)^2 = 8 - 6i$ $(x + iy)(x + iy) = 8 - 6i$ $x^2 + 2xyi - y^2 = 8 - 6i$ $(x^2 - y^2) + i(2xy) = 8 - 6i$ Equating real and imaginary parts gives: $x^2 - y^2 = 8$ and $2xy = -6$ Solving simultaneously gives: $x = \pm 3, y = \mp 1,$ $\Rightarrow \mathbf{z_1 = 3 - i}$ **and** $\mathbf{z_2 = -3 + i}$

Real Versus Imaginary

$\boldsymbol{Re(z)}$ **means the real part of z and $\boldsymbol{Im(z)}$ means the imaginary part of z**

- If a number is **real only** it just means the imaginary part is 0 : 2 is the same as saying $2 + 0i$
- If a number is **imaginary only** you will just see an i term : $3i$ is the same as saying $0 + 3i$

Proving purely real versus purely imaginary

- **A real number (purely real)** means only consider the real part of the complex number and set the imaginary part equal to zero.
- An **imaginary number (purely imaginary)** means only consider the imaginary part of the complex number and set the real part equal to zero.

Example 1	Example 2
State the real and imaginary parts of $z = 2 - 3i$	Find the value of a if $z = 2 - 2a + (a + 4)i$ is purely imaginary
$Re(z) = \mathbf{2}$ $Im(z) = \mathbf{-3}$	$Re(z) = 0$ $2 - 2a = 0$ $\mathbf{a = 1}$

Factors and Roots

Factorising And Solving Quadratics, Cubics and Quartics

Step 1: Find a factor (usually given a factor). The complex conjugate is also always a factor. Once we know 2 factors, build $z^2 -$ (sum roots)$z +$ product roots

Step 2: To find the remaining factor(s), we can then use either

- **Algebraic division** - divide the polynomial by the factor found above
- **Compare coefficients** – write as factor times a general quadratic or cubic, expand and set coefficients equal

Example 1: Solving a Quadratic (warm up)	Example 2: Factorising a Cubic	Example 3: Factorising A Quartic
Solve $z^2 + 4z + 8 = 0$	$(z + 2i)$ is a factor of $2z^3 - 3z^2 + 8z - 12$. Find the other two factors and fully factorise	Factor $x^4 - 5x^3 + 2x^2 + 22x - 20$ completely, given $3 - i$ is a root
Using quadratic formula $1z^2 + 4z + 8 = 0$ has roots $z = \frac{-4 \pm \sqrt{4^2 - 4(1)(8)}}{2(1)}$ $z = \frac{-4 \pm \sqrt{-16}}{2}$ $z = \frac{-4 \pm 4i}{2}$ $\mathbf{z = -2 \pm 2i}$	$(z + 2i)$ is a factor and so $-2i$ is a root The complex conjugate $+2i$ is also a root $z^2 -$ (sum roots)$z +$ product roots Sum of roots $= 2i + (-2i) = 0$ Product of roots $= 2i(-2i) = -4i^2 = 4$ $z^2 - 0z + 4$ Long division of $2z^3 - 3z^2 + 8z - 12$ by $z^2 + 4$ gives $2z - 3$, remainder 0 The other two factors are $(z - 2i)$ and $(2z - 3)$ Fully factorised $2z^3 - 3z^2 + 8z - 12$ looks like $(z^2 + 4)(2z - 3)$	$3 - i$ is a root $\Rightarrow$ conjugate $3 + i$ is also a root $x^2 -$ (sum roots)$x +$ product roots Sum of roots $= 3 - i + 3 + i = 6$ Product of roots $= (3 - i)(3 + i) = 10$ $x^2 - 6x + 10$ Long division of $x^4 - 5x^3 + 2x^2 + 22x - 20$ by $x^2 - 6x + 10$ gives $x^2 + x - 2$, remainder 0 Hence, we can write $x^4 - 5x^3 + 2x^2 + 22x - 20$ $(x^2 - 6x + 10)(x^2 + x - 2)$ $= (x^2 - 6x + 10)(x - 1)(x + 2)$

Example 4: Solving A Cubic	Example 5: Solving A Quartic	Example 6: Solving A Quartic
The complex number $x = 1 + 2i$ is a solution of the equation $x^3 - 3x^2 + 7x - 5 = 0$. Find the other solutions.	Find all complex numbers z, such that: $z^4 - z^3 + 6z^2 - z + 15 = 0$ and $z = 1 + 2i$ is a solution	Given that $(z - 1 - 2i)$ is a factor of $2z^3 - 3z^2 + 8z + 5$ solve the equation $2z^3 - 3z^2 + 8z + 5 = 0$
$1 - 2i$ is a root $\Rightarrow$ $1 + 2i$ is also a root Sum of roots $= 1 + 2i + 1 - 2i = 2$ Product of roots $= (1 + 2i)(1 - 2i) = 5$ $z^2 -$ (sum roots)$z +$ product roots $= x^2 - 2x + 5$ Long division of $x^3 - 3x^2 + 7x - 5$ by $x^2 - 2x + 5$ gives $x - 1$, remainder 0 $= (x^2 - 2x + 5)(x - 1)$	$x^2 -$ (sum roots)$x +$ product roots Sum of roots $= 1 + 2i + 1 - 2i = 2$ Product of roots $= (1 + 2i)(1 - 2i) = 5$ $x^2 - 2x + 5$ Long division of $x^4 - x^3 + 6x^2 - x + 15$ by $x^2 - 2x + 5$ gives $x^2 + x + 3$, remainder 0 Hence, we can write $x^4 - x^3 + 6x^2 - x + 15$ as $(x^2 - 2x + 5)(x^2 + x + 3)$	$(z - (1 + 2i))$ is a factor means $1 + 2i$ is a root $z^2 -$ (sum roots)$x +$ product roots Sum of roots $= 1 + 2i + (1 - 2i) = 2$ Product of roots $= (1 + 2i)(1 - 2i) = 5$ $x^2 - 2x + 5$ Long division of $2z^3 - 3z^2 + 8z + 5$ by $z^2 - 2z + 5$ gives $2z + 1$, remainder 0 $2z + 1 = 0 \Rightarrow z = -\frac{1}{2}$ The other 2 solutions are $1 - 2i$ and $-\frac{1}{2}$

Building Equations/Finding UNKNOWN Coefficients

Cubics:

- A cubic with real coefficients either has
  - All three roots real:
    $(x - \text{real root 1})(x^2 - (\text{sum of real roots 2 and 3})x + (\text{product of real roots 2 and 3}))$
  - One root real and the other two form a complex conjugate pair:
    $(x - \text{real root})(x^2 - (\text{sum of complex conjugate roots})x + (\text{product of complex conjugate roots}))$

Quartics:

- A quartic with real coefficients either has:
  - All four roots real:
    $(x^2 - (\text{sum of real roots})x + (\text{product of real roots}))(x^2 - (\text{sum of real roots})x + (\text{product of real roots}))$
  - Two roots real and the other two form a complex conjugate pair:
    $(x^2 - (\text{sum of real roots})x + (\text{product of real roots}))(x^2 - (\text{sum of complex roots})x + (\text{product of complex roots}))$
  - Two roots form a complex conjugate pair, and the other two roots also form a complex conjugate pair:
    $(x^2 - (\text{sum of complex roots})x + (\text{product of complex roots}))(x^2 - (\text{sum of complex roots})x + (\text{product of complex roots}))$

Example 1	Example 2	Example 3	Example 4
Find a third-degree polynomial with integer coefficients that has zeros $2 - 3i$ and 1	Given that $2 - i$ is a root of the equation, $x^3 + ax^2 - 3x + b = 0$, where the constants a and b are real. Find the values of a and b	$3z^3 + pz^2 + 57z + q$ where p and q are real constants. Given that $3 - 2\sqrt{2}i$ is a root of the equation, find all the roots	Consider $x^4 + ax^3 + bx^2 + cx + d$, where $a, b, c \in \mathbb{R}$. Given that $1 + i$ and $1 - 2i$ are zeros find the values of a, b, c, and d
Zero 1 $2 - 3i \Rightarrow 2 + 3i$ is a root sum $= 2 - 3i + 2 + 3i = 4$ Prod $= (2 - 3i)(2 + 3i) = 13$ $= (x^2 - 4x + 13)(x - 1)$	$2 - i \Rightarrow$ conjugate $2 + i$ is also a root of the equation Sum of roots $= 2 - i + (2 + i) = 4$ Product of roots $= (2 - i)(2 + i) = 5$ $x^2 - 4x + 5$ $(x^2 - 4x + 5)(x - d) = x^3 + ax^2 - 3x + b$ LHS: $x^3 - 4x^2 + 5x - dx^2 + 4dx - 5d$ $x^3 + (-4 - d)x^2 + (5 + 4d)x - 5d$ Compare coefficients with $x^3 + ax^2 - 3x + b$ $5 + 4d = -3$ $4d = -8$ $d = -2$ $-4 - d = a$ $-4 + 2 = a$ $a = -2$ $b = -5d$	$3 - 2\sqrt{2}i \Rightarrow 3 + 2\sqrt{2}i$ is also a root Sum of roots $= 6$ Product of roots $= 17$ $(z^2 - 6z + 17)(az + b) = 3z^3 + pz^2 + 57z + q$ LHS: $(z^2 - 6z + 17)(az + b)$ $az^3 + bz^2 - 6az^2 - 6bz + 17az + 17b$ $= az^3 + (b - 6a)z^2 + (-6b + 17a)z + 17b$ Compare coefficients with $3z^3 + pz^2 + 57z + q$ Hence, we have $(z^2 - 6z + 17)(3z - 1)$ The third root is $\frac{1}{3}$	$1 + i$ and $1 - 2i$ are roots $\Rightarrow 1 - i$ and $1 + 2i$ are roots Sum of roots $= 2$ and 2 Product of roots $= 2$ and 5 $(x^2 - 2x + 2)(x^2 - 2x + 5) = x^4 + ax^3 + bx^2 + cx + d$ LHS: $(x^2 - 2x + 2)(x^2 - 2x + 5)$ $= x^4 - 4x^3 + 11x^2 - 14x + 10$ Comparing the coefficients with $x^4 + ax^3 + bx^2 + cx + d$ $a = -4$ $b = 11$ $c = -14$ Finally, comparing the constant term: $d = 10$

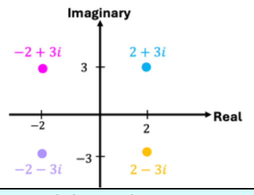
Faster method (using polynomial root chapter): Type the roots into calc (store each as a letter and then you can find the sum, sum of all possible products and products quickly by typing these in letter-wise

$x^3 - (\text{sum of roots})x^2 + (\text{sum of all possible products of roots})x - (\text{product of roots})$

$x^4 - (\text{sum of roots})x^3 + (\text{sum of all possible products of pairs roots})x^2 - (\text{sum of all possible products of triples of roots})x + (\text{product of roots})$

Argand Diagram

Representation On An Argand Diagram



The real part goes on the x axis and the imaginary part goes on the y axis.

- $2 + 3i$
- $-2 + 3i$
- $-2 - 3i$
- $2 - 3i$

Modulus and Argument

Complex numbers have a **length (modulus)** and an **angle (argument)**

- **Modulus** = length of complex number which is denoted by $|\ |$
- **Argument** = angle (θ) between the **positive real axis** and the line joining the complex number to the origin

How to find the Modulus and Argument:

4 Examples

Draw the complex number in the correct quadrant, form a triangle and use Pythagoras for the magnitude and SOHCAHTOA for the argument (remember you can't do further than π or $-\pi$ for the argument.

Example 1	Example 2	Example 3	Example 4
$2 + 3i$	$-2 + 3i$	$2 - 3i$	$-2 - 3i$
Form triangle with the x axis	Form triangle with the x axis	Form triangle with the x axis	Form triangle with the x axis
modulus length $= \sqrt{2^2 + 3^2} = \sqrt{13}$	modulus length $= \sqrt{2^2 + 3^2} = \sqrt{13}$	modulus length $= \sqrt{2^2 + 3^2} = \sqrt{13}$	modulus length $= \sqrt{2^2 + 3^2} = \sqrt{13}$
argument $\tan\theta = \frac{3}{2}$ arg $= 56.3°$	argument $\tan\theta = \frac{3}{2}$ arg $= 180 - 56.3° = 123.7°$	argument $\tan\theta = \frac{3}{2}$ arg $= -56.3°$	argument $\tan\theta = \frac{3}{2}$ arg $= -180 + 56.3° = -123.7°$

Template Formula (read this after the examples)

Given form $z = x + yi$

To Find The Modulus $= |z|$:

$$\sqrt{x^2 + y^2}$$

To find the argument $= \arg(z)$:

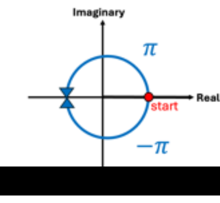
Step 1:

$$\tan^{-1}\left(\left|\frac{y}{x}\right|\right) \text{ where } -\pi < \theta < \pi$$

Note: The green parts are always a plus

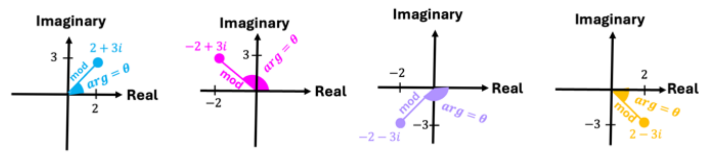
Step 2: Draw $x + yi$ out to know which quadrant you're in.

Step 3: Read off the angle. To do this start from positive x axis and find the anticlockwise angle to find the value of theta. Remember the limits when finding the angle. The argument angle is always between $-\pi$ and π. This means we start on the x axis and can't go further than π in either direction (pos or neg)

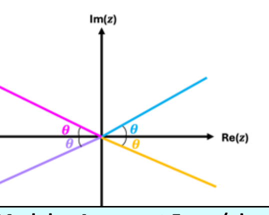


Summary Diagrams

Summary Diagrams:

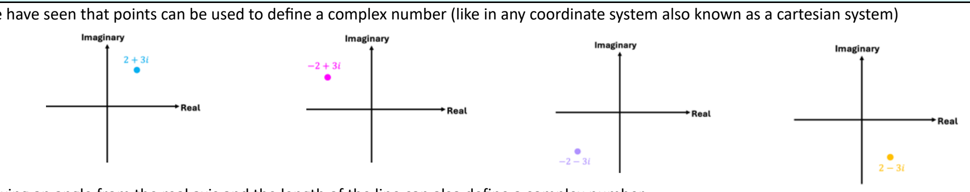


In one diagram this looks like:

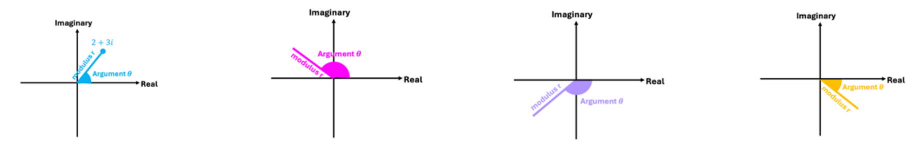


Modulus Argument Form (aka polar form)

So far we have seen that points can be used to define a complex number (like in any coordinate system also known as a cartesian system)



And knowing an angle from the real axis and the length of the line can also define a complex number



Modulus form does the following (turns cartesian into a modulus form)

$$x + yi \rightarrow r(\cos\theta + i\sin\theta)$$

Method To Convert Cartesian To Modulus Argument Form:

We just need to find r and θ. To find r and θ using the following

$$x + yi \Rightarrow \begin{cases} r = \text{modulus} = \sqrt{x^2 + y^2} \\ \theta = \text{argument} = \tan^{-1}\left(\left|\frac{y}{x}\right|\right) \end{cases}$$

Fill r and θ back into the form $r(\cos\theta + i\sin\theta)$

Method To Convert Modulus Argument Form To Cartesian:

$$r(\cos(\theta) + i\sin(\theta))$$

multiply out

$$= r\cos(\theta) + ir\sin(\theta)$$

Then work out each trig value as a number

Negative angles (odd versus even)

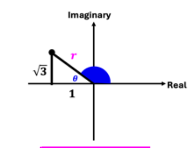
$$\cos(-\theta) + i\sin(-\theta) = \cos(\theta) - i\sin(\theta)$$

Complex Conjugate:

$$z^* = r(\cos\theta - i\sin\theta)$$

Example 1: Converting Cartesian to Modulus Argument Form

Turn into $-1 + \sqrt{3}i$ into modulus-argument form



Modulus $= r = \sqrt{(-1)^2 + (\sqrt{3})^2} = \sqrt{4} = 2$

Let's draw for the modulus

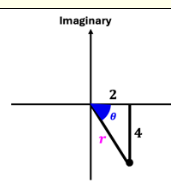
$\theta = \tan^{-1}\left(\frac{\sqrt{3}}{1}\right) = \frac{\pi}{3}$

Argument $= \frac{2\pi}{3}$

Plugging this into the form we get

$$2\left(\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right)$$

Turn into $2 - 4i$ into modulus-argument form



Modulus $= r = \sqrt{(2)^2 + (-4)^2} = 2\sqrt{5}$

Let's draw for the modulus

$\theta = \tan^{-1}\left(\frac{4}{2}\right) = 1.11$

Argument $= -1.11$

Plugging this into the form we get

$= 2\sqrt{5}(\cos(-1.11) + i\sin(-1.11))$

$= 2\sqrt{5}(\cos(1.11) - i\sin(-1.11))$

Example 2: Converting Modulus Argument Form To Cartesian

Turn $4\left(2\cos\frac{\pi}{2} - i\sin\frac{\pi}{2}\right)$ into Cartesian form

$$4\left(2\cos\frac{\pi}{2} - i\sin\frac{\pi}{2}\right)$$

$$4 \times 2\cos\frac{\pi}{2} - 4 \times i\sin\frac{\pi}{2}$$

$$= 8\cos\frac{\pi}{2} - 4i\sin\frac{\pi}{2}$$

Now work out each trig part

$$= 8\cos\frac{\pi}{2} - 4i\sin\frac{\pi}{2}$$

$$= 8(0) - 4i(1)$$

$$= -4i$$

Multiplying and Dividing Properties

Modulus Results:

- $|z_1 z_2| = |z_1||z_2|$
- $\left|\frac{z_1}{z_2}\right| = \frac{z_1}{z_2}$

Argument Results:

- $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$. This works like indices rules. When we multiply, we add the powers.
- $\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$. This works like indices rules. When we divide. we subtract the powers

Multiplying Modulus Argument Form:

$$[r_1(\cos\theta_1 + i\sin\theta_1)][r_2(\cos\theta_2 + i\sin\theta_2)] = r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$$

Hence multiplying 2 complex numbers is the same as multiplying their lengths and adding their angles.

It should also make sense that power n means raise r to power n but multiply angles by n (you'll learn De Moivre's in year 2)

Dividing Modulus Argument Form:

$$\frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]$$

Hence dividing 2 complex numbers is the same as dividing their length and subtracting their angles

Example 1: Mod/Arg of Two Multiplied Complex Numbers

$z_1 = 3\left(\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right)$

$z_2 = \sqrt{2}\left(\cos\left(\frac{\pi}{12}\right) - i\sin\left(\frac{\pi}{12}\right)\right)$

Write down the exact value of

i. $|z_1 z_2|$

ii. $\arg(z_1 z_2)$

$$z_1 = 3\left(\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right)$$

$$z_2 = \sqrt{2}\left(\cos\left(\frac{\pi}{12}\right) - i\sin\left(\frac{\pi}{12}\right)\right)$$

$$= \sqrt{2}\left(\cos\left(-\frac{\pi}{12}\right) + i\sin\left(-\frac{\pi}{12}\right)\right)$$

i.

$|z_1| = 3$ and $|z_2| = \sqrt{2}$

$|z_1 z_2| = |z_1||z_2| = 3\sqrt{2}$

ii.

$\arg(z_1) = \frac{\pi}{3}$ and $\arg(z_2) = -\frac{\pi}{12}$

$\arg(z_1 z_2) = \arg(z_1) + \arg(z_2) = \frac{\pi}{3} - \frac{\pi}{12} = \frac{\pi}{4}$

Example 2: Multiplying and Dividing in Modulus Argument Form

$z = a + ib$, where a and b are real constants

i. $|z^{10}| = 59049$

ii. $\arg(z^{10}) = -\frac{5\pi}{3}$

Determine the value of a and the value of b

If we know the mod and argument we can find z. Let's find this

i.

$|z^{10}| = 59049$ hence $|z|^{10} = 59049$

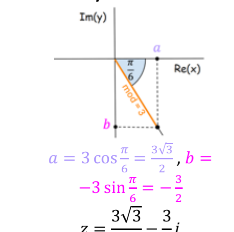
$|z| = \sqrt[10]{59049} = 3$

ii.

$\arg(z^{10}) = -\frac{5\pi}{3}$

$10\arg z = -\frac{5\pi}{3}$

$\arg z = -\frac{\pi}{6}$ hence $\tan\left(-\frac{\pi}{6}\right) = z$

Way 1: plug into mod arg form	Way 2: Draw out
$r\cos(\theta + i\sin\theta)$ $3\cos\left(\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right)$ $z = \frac{3\sqrt{3}}{2} - \frac{3}{2}i$	 $a = 3\cos\frac{\pi}{6} = \frac{3\sqrt{3}}{2}$, $b = -3\sin\frac{\pi}{6} = -\frac{3}{2}$ $z = \frac{3\sqrt{3}}{2} - \frac{3}{2}i$

Loci and Regions Continued

Type 1: Modulus = number gives a circle

$|z - z_1| = k \Rightarrow$ **circle** centre z_1 and radius k

Proof:

$|(x + iy) - (a + ib)| = k$ — Replace z with $x + iy$; Replace z_1 with $a + ib$

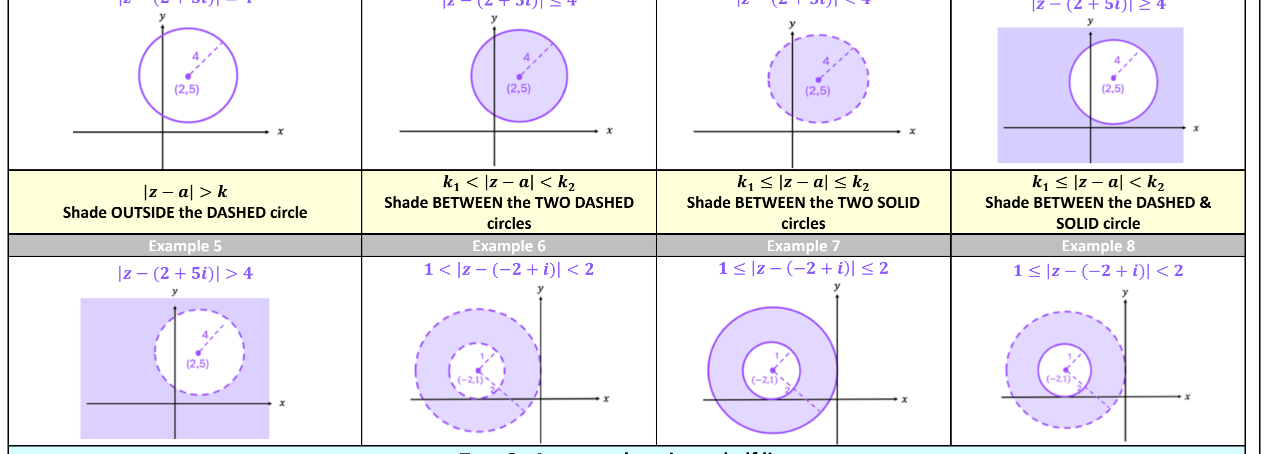
$|(x - a) + i(y - b)| = k$

$\sqrt{(x - a)^2 + (y - b)^2} = k$ — Applying the definition of modulus

$(x - a)^2 + (y - b)^2 = k^2$ — Use algebra to re-arrange

This is the **CARTESIAN** equation you get (it will be a circle)

$\lvert z - a\rvert = k$ A circle centre 'a' radius k	$\lvert z - a\rvert \le k$ Shade INSIDE the SOLID circle	$\lvert z - a\rvert < k$ Shade INSIDE the DASHED circle	$\lvert z - a\rvert \ge k$ Shade OUTSIDE the SOLID circle
Example 1: $\lvert z - (2 + 5i)\rvert = 4$	Example 2: $\lvert z - (2 + 5i)\rvert \le 4$	Example 3: $\lvert z - (2 + 5i)\rvert < 4$	Example 4: $\lvert z - (2 + 5i)\rvert \ge 4$
$\lvert z - a\rvert > k$ Shade OUTSIDE the DASHED circle	**$k_1 < \lvert z - a\rvert < k_2$ Shade BETWEEN the TWO DASHED circles**	**$k_1 \le \lvert z - a\rvert \le k_2$ Shade BETWEEN the TWO SOLID circles**	**$k_1 \le \lvert z - a\rvert < k_2$ Shade BETWEEN the DASHED & SOLID circle**
Example 5: $\lvert z - (2 + 5i)\rvert > 4$	Example 6: $1 < \lvert z - (-2 + i)\rvert < 2$	Example 7: $1 \le \lvert z - (-2 + i)\rvert \le 2$	Example 8: $1 \le \lvert z - (-2 + i)\rvert < 2$



Type 2: Arg = number gives a half line

$\boldsymbol{\arg(z - z_1) = \theta}$ **is a** <u>**half line**</u> from but <u>not including</u> the fixed point z_1 making an **anticlockwise** angle θ with a line from the fixed point z_1 **parallel to the real axis**

Proof:

$\arg(z - z_1) = \theta$ — Replace z with $x + iy$; Replace z_1 with $a + ib$

$\arg((x + iy) - (a + ib)) = \theta$

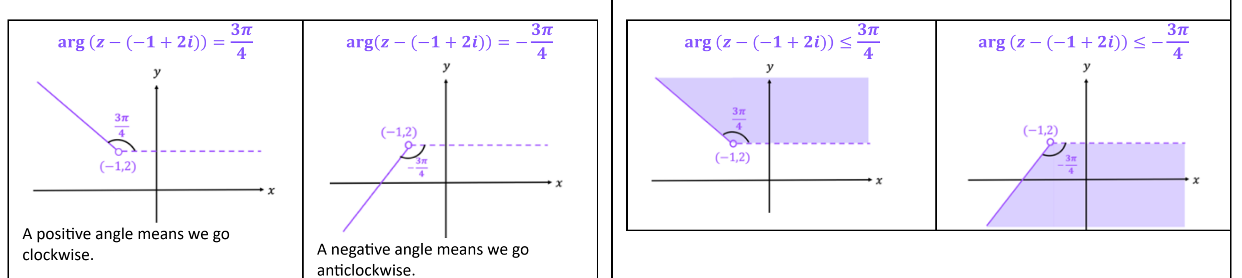
$\arg((x - a) + i(y - b)) = \theta$ — Apply the definition of argument

$\tan(\theta) = \frac{y - b}{x - a}$ — Use algebra to re-arrange

$y - b = \tan(\theta)(x - a)$

θ is a fixed angle so $\tan(\theta)$ is a constant hence this is the equation of a straight line (half line) with gradient $\tan(\theta)$ passing through (a, b)

$\arg(z - a) = \beta$	$\arg(z - a) \le \beta$ or $\arg(z - a) \le -\beta$
half line starting at point 'a' (an undefined point ∘) at an angle of β where $-\pi < \beta < \pi$. The angle is anticlockwise or clockwise from the positive real axis (draw a dashed line parallel to x axis starting from the point) Note: we can have $0 < \beta < 2\pi$, but it's less common **e 1**	**Half line starting at point 'a' going clockwise at an angle less than or equal to the angle β or $-\beta$**
Example 1	Example 2 Example 1
$\arg(z - (-1 + 2i)) = \frac{3\pi}{4}$ — A positive angle means we go clockwise. $\arg(z - (-1 + 2i)) = -\frac{3\pi}{4}$ — A negative angle means we go anticlockwise.	$\arg(z - (-1 + 2i)) \le \frac{3\pi}{4}$ $\arg(z - (-1 + 2i)) \le -\frac{3\pi}{4}$

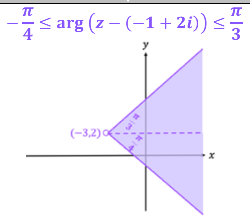


$-\beta \le \arg(z - a) \le \gamma$

Half line starting at 'a' between an angle of β anticlockwise and γ clockwise

Example 1

$$-\frac{\pi}{4} \le \arg(z - (-1 + 2i)) \le \frac{\pi}{3}$$



Tip: choose a point in the region and see if it satisfies the inequality.

Type 3: Modulus = Modulus gives a perpendicular bisector (straight line)

$|z - z_1| = |z - z_2|$ **is the perpendicular bisector** of the line segment joining the points z_1 and z_2. Notice how there is a modulus on either side

Proof:

$|(x + iy) - (a + ib)| = |(x + iy) - (c + id)|$ — Replace z with $x + iy$; Replace z_1 with $a + ib$; Replace z_2 with $c + id$

$|(x - a) + i(y - b)| = |(x - c) + i(y - d)|$

$\sqrt{(x - a)^2 + (y - b)^2} = \sqrt{(x - c)^2 + (y - d)^2}$ — Applying the definition of modulus

$(x - a)^2 + (y - b)^2 = (x - c)^2 + (y - d)^2$ — Use algebra to re-arrange

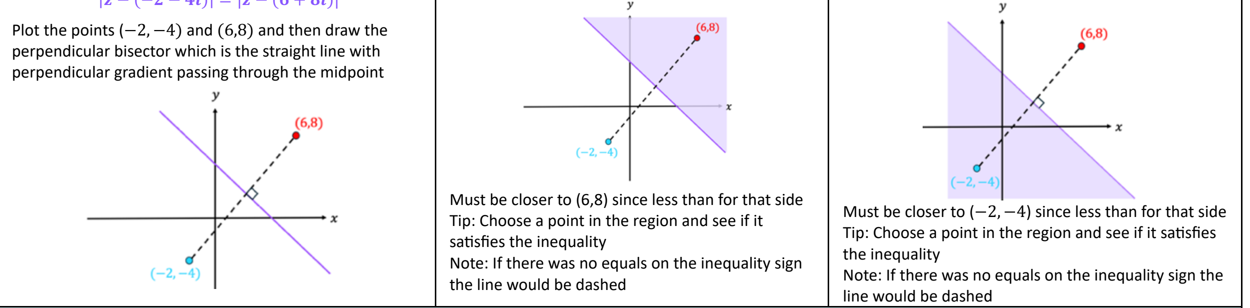
Multiply out and re-arrange (the quadratic x^2 and y^2 terms cancel out)

$$-2ax - 2by = -2cx - 2dx$$

This is just a straight line with gradient perpendicular to the gradient of the line segment joining the two points and going through the midpoint of this line segment.

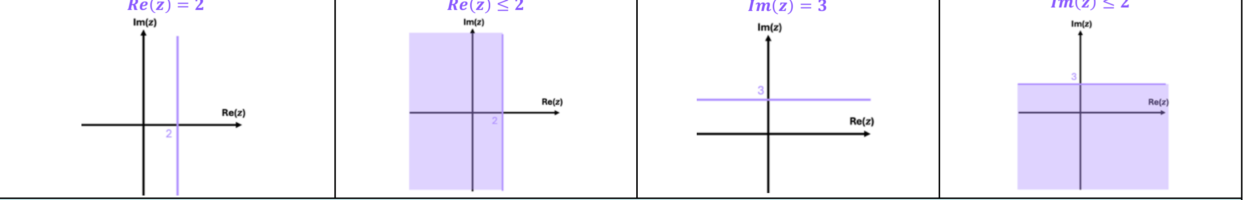
Note: You're usually given the vectors z_1 and z_2, so you won't have these letters a, b, c, d above.

$\lvert z - a\rvert = \lvert z - b\rvert$ A straight line - Perpendicular bisector of the line joining the the points a and b	$\lvert z - a\rvert \ge \lvert z - b\rvert$ The distance of z from a must be greater than or equal to the distance of z from b	$\lvert z - a\rvert \le \lvert z - b\rvert$ The distance of z from a must be less than or equal to the distance of z from b
Example 1	Example 2	Example 3
$\lvert z - (-2 - 4i)\rvert = \lvert z - (6 + 8i)\rvert$ Plot the points $(-2, -4)$ and $(6,8)$ and then draw the perpendicular bisector which is the straight line with perpendicular gradient passing through the midpoint	$\lvert z - (-2 - 4i)\rvert \ge \lvert z - (6 + 8i)\rvert$ Must be closer to (6,8) since less than for that side Tip: Choose a point in the region and see if it satisfies the inequality Note: If there was no equals on the inequality sign the line would be dashed	$\lvert z - (-2 - 4i)\rvert \le \lvert z - (6 + 8i)\rvert$ Must be closer to $(-2, -4)$ since less than for that side Tip: Choose a point in the region and see if it satisfies the inequality Note: If there was no equals on the inequality sign the line would be dashed



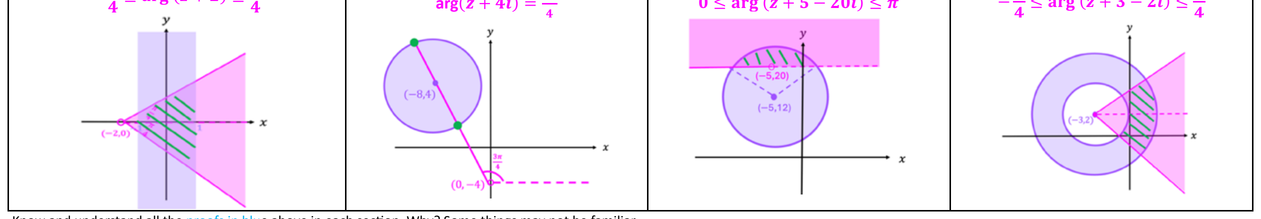
Type 4: Re(z) and Im(z) are vertical and horizontal lines

$\text{Re}(z)$ = Number is a vertical line: $\text{Re}(x + iy) = a$ is the line $x = a$		$\text{Im}(z)$ = Number is a horizontal line: $\text{Im}(x + iy) = b$ is the line $y = b$	
Example 1	Example 3	Example 2	Example 4
$Re(z) = 2$	$Re(z) \le 2$	$Im(z) = 3$	$Im(z) \le 2$



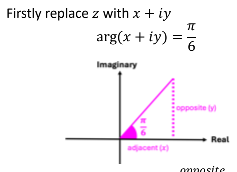
Type 5: Intersections (shade both and find the common/overlap region)

Example 1	Example 2	Example 3	Example 4
$-1 < Re(z) \le 1$ $-\frac{\pi}{4} \le \arg(z + 2) \le \frac{\pi}{4}$	$\lvert z + 8 - 4i\rvert = 2$ $\arg(z + 4i) = \frac{3\pi}{4}$	$\lvert z + 5 - 12i\rvert \le 10$ $0 \le \arg(z + 5 - 20i) \le \pi$	$3 \le \lvert z + 3 - 2i\rvert \le 5$ $-\frac{\pi}{4} \le \arg(z + 3 - 2i) \le \frac{\pi}{4}$



Know and understand all the proofs in blue above in each section. Why? Some things may not be familiar

Cartesian Equations And Unfamiliar Regions

Example 1:	Example 2:	Example 3: Unfamiliar region	Example 4: Unfamiliar region
Find the cartesian equation of the locus of $\lvert z - 3\rvert = \lvert z + i\rvert$	Find the cartesian equation of the locus of $\arg z = \frac{\pi}{6}$	Find the cartesian equation of the locus of $\lvert z - 3\rvert = 2\lvert z + i\rvert$	Find the cartesian equation of the locus of $\lvert z + 2i\rvert = \lvert 2iz - 1\rvert$
$\lvert z - 3\rvert = \lvert z + i\rvert$ Firstly replace z with $x + iy$ $\lvert x + iy - 3\rvert = \lvert x + iy + i\rvert$ Group the real and imaginary parts $\lvert x - 3 + iy\rvert = \lvert x + i(y + 1)\rvert$ Apply the modulus definition $\sqrt{(x - 3)^2 + (y)^2} = \sqrt{(x)^2 + (y + 1)^2}$ $x^2 - 6x + 9 + y^2 = x^2 + y^2 + 2y + 1$ $-6x + 8 = 2y$ $y = -3x + 4$ This is a **straight line**	Firstly replace z with $x + iy$ $\arg(x + iy) = \frac{\pi}{6}$  SOHCAHTOA says $\tan\theta = \frac{opposite}{adjacent}$ $\tan\frac{\pi}{6} = \frac{y}{x}$ $\frac{1}{\sqrt{3}} = \frac{y}{x}$ $y = \frac{1}{\sqrt{3}}x$ (**half line**) Hindsight: The equation of a half line is just $y - b = \tan\theta(x - a)$ where (a,b) is the undefined point the half line starts from and θ is the argument	Notice how we have a number outside the modulus hence not a familiar circle Firstly replace z with $x + iy$ $\lvert x + iy - 3\rvert = 2\lvert x + iy + 1i\rvert$ Group the real and imaginary parts $\lvert x - 3 + iy\rvert = 2\lvert x + iy + 1i\rvert$ Apply the modulus definition $\sqrt{(x - 3)^2 + y^2} = 2\sqrt{x^2 + (y + 1)^2}$ $x^2 - 6x + 9 + y^2$ $= 4x^2 + 4(y^2 + 2y + 1)$ Expand the brackets $x^2 - 6x + 9 + y^2 = 4x^2 + 4y^2 + 8y + 4$ Group common terms $3x^2 + 6x + 3y^2 + 8y = 5$ Completing the square $3(x + 1)^2 - 3 + 3\left(y + \frac{4}{3}\right)^2 - \frac{16}{3} = 5$ $(x + 1)^2 + \left(y + \frac{4}{3}\right)^2 = \frac{40}{9}$ This corresponds to a **circle** centre $\left(-1, -\frac{4}{3}\right)$ radius $= \sqrt{\frac{40}{9}} = \frac{2\sqrt{10}}{3}$	Notice how we have a number inside the modulus next to z hence not a familiar circle Firstly replace z with $x + iy$ $\lvert x + iy + 2i\rvert = \lvert 2i(x + iy) - 1\rvert$ Group the real and imaginary parts $\lvert x + i(y + 2)\rvert = \lvert 2xi - 2y - 1\rvert$ Apply the modulus definition $\sqrt{x^2 + (y + 2)^2} = \sqrt{(-2y - 1)^2 + (2x)^2}$ $\sqrt{x^2 + y^2 + 4y + 4} = \sqrt{4y^2 + 4y + 1 + 4x^2}$ $x^2 + y^2 + 4y + 4 = 4y^2 + 4y + 1 + 4x^2$ $3x^2 + 3y^2 - 3 = 0$ $3x^2 + 3y^2 = 3$ $x^2 + y^2 = 1$ This corresponds to a **circle** (0,0) radius 1

Loci and Regions Continued

Min/Max Distances

$|z|$ represents the distance from the origin

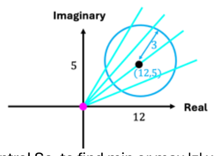
Example 1: With A Circle

Given that the complex number $z = x + iy$ satisfies the equation $|z - 12 - 5i| = 3$, find the minimum value of $|z|$ and the maximum value of $|z|$

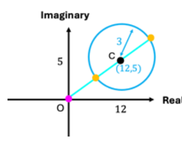
$|z - 12 - 5i| = 3$ is a circle centre (12,5) radius 3

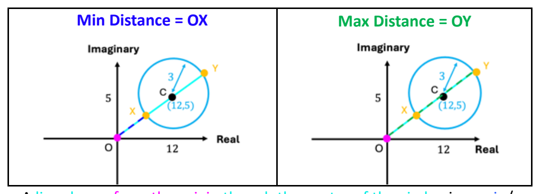
$|z|$ represents the distance from the origin

Which line gives the **shortest and largest distance** on the circle from the origin?



It is the line though the centre! So, to find min or max |z| we always draw a line from the origin through the centre.

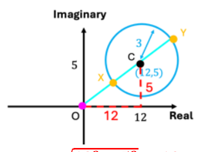




A line drawn from the origin though the centre of the circle gives min/max |z|
The min/max distance from the origin to the circle which is OX or OY.

Way 1: Geometric

Let's build a right angled triangle to find OC (or use distance formula with the points (0,0) and (2,5))



$OC = \sqrt{12^2 + 5^2} = 13$

Now we use the fact that we know the radius length to help us

Min distance/value $= OX = OC - XC = 13 - 3 = 10$

Max distance/value $= OY = OC + CY = 13 + 3 = 16$

Way 2: Algebraic

Equation of the line is $y = \frac{5}{12}x$

Now we can solve $y = \frac{5}{12}x$ and $(x-12)^2 + (y-5)^2 = 9$ simultaneously to find the orange points

The coordinates are $X = \left(\frac{120}{13}, \frac{50}{13}\right)$ and $Y = \left(\frac{192}{13}, \frac{80}{13}\right)$

Min distance/value $= OX = \sqrt{\left(\frac{120}{13}\right)^2 + \left(\frac{50}{13}\right)^2} = 10$

Max distance/value $= OY = \sqrt{\left(\frac{192}{13}\right)^2 + \left(\frac{80}{13}\right)^2} = 16$

Example 2: With A Circle (circle goes over the origin)

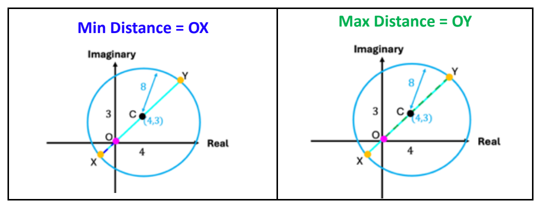
$|z - 4 - 3i| = 8$

Find the minimum value of $|z|$ and the maximum value of $|z|$

$|z - 4 - 3i| = 8$

The previous example explained why a line drawn from the origin though the centre of the circle gives min/max |z|

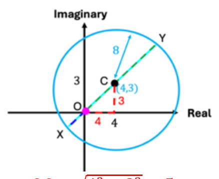
The min/max distance from the origin to the circle which is OX or OY.



The min/max distance from the origin to the circle which is OX or OY.

Way 1: Geometric

Let's build a right angled triangle to find OC (or use distance formula with the points (0,0) and (4,3))



$OC = \sqrt{4^2 + 3^2} = 5$

Now we use the fact that we know the radius length to help us

Min distance/value $= OX = CX - OC = 8 - 5 = 3$

Max distance/value $= OY = CX + OC = 8 + 5 = 13$

Way 2: Algebraic

Equation of the line is $y = \frac{3}{4}x$

We can solve $y = \frac{3}{4}x$ and $(x-4)^2 + (y-3)^2 = 64$ simultaneously to find the orange points

$x = \frac{52}{5}, y = \frac{39}{5}$ or $x = -\frac{12}{5}, y = -\frac{9}{5}$

The coordinates are $X = \left(\frac{52}{5}, \frac{39}{5}\right)$ and $Y = \left(-\frac{12}{5}, -\frac{9}{5}\right)$

Min distance/value $= OX = \sqrt{\left(\frac{52}{5}\right)^2 + \left(\frac{39}{5}\right)^2} = 13$

Max distance/value $= OY = \sqrt{\left(-\frac{12}{5}\right)^2 + \left(-\frac{9}{5}\right)^2} = 3$

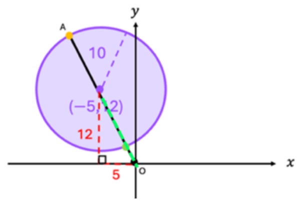
Example 3: With A Circle

$|z + 5 - 12i| \le 10$

For the set of points defined, determine the maximum value of |z|

Note: The inequality makes no difference. It is just saying the region within or on the circle, but we know the min and max will still be the points on the circumference of the circle

$|z + 5 - 12i| \le 10$



|z| is the distance from the ORIGIN

The maximum value of |z| is the length of the line segment going from **the origin** all the way to the furthest point in the loci. In this case, the maximum value of |z| is the length of OA, as seen in the diagram.

To find this maximum value, we can just calculate the distance between the centre of the circle and the origin and add on the radius of the circle to this value.

Distance from origin to centre of circle $= \sqrt{12^2 + 5^2} = 13$

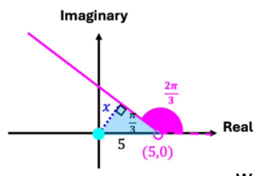
$OA = \text{maximum of } |z| = 13 + 10 = 23$

Example 4: With Argument

Given that $arg(z - 5) = \frac{2\pi}{3}$. Determine the least value of $|z|$ as z varies

Note: This is the same as just saying find the minimum value of |z|

Let's first draw a diagram



$|z|$ represents the distance from the origin to the half line which is x.

The **minimum distance** is always the **perpendicular** distance x hence we form a right angled triangle

Way 1: Geometric

We know the angle $\frac{\pi}{6}$ so we can build a triangle and use Pythagoras to find the length x

$$\sin\frac{\pi}{3} = \frac{x}{5}$$

$$x = \frac{5\sqrt{3}}{2}$$

Way 2: Algebraic

Way 1:	Way 2: Use template $y - b = \tan\theta(x - a)$
$arg(z-5) = \frac{2\pi}{3}$ $arg(x + iy - 5) = \frac{2\pi}{3}$ $\tan\frac{2\pi}{3} = \frac{y}{x-5}$ $-\sqrt{3} = \frac{y}{x-5}$ $y = -\sqrt{3}x + 5\sqrt{3}$	$y - 0 = \tan\frac{2\pi}{3}(x-5)$ $y = -\sqrt{3}(x-5)$ $y = -\sqrt{3}x + 5\sqrt{3}$

The blue line is perpendicular since want the shortest distance hence $y = \frac{1}{\sqrt{3}}x + c$

Line passes through the origin hence $y = \frac{1}{\sqrt{3}}x$

Solving $y = -\sqrt{3}x + 5\sqrt{3}$ and $y = \frac{1}{\sqrt{3}}x$ simultaneously gives $x = \frac{15}{4}$

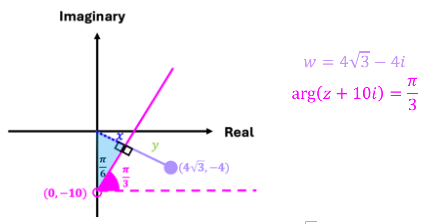
The intersection point is $\left(\frac{15}{4}, \frac{5\sqrt{3}}{4}\right)$

Distance from $(0,0)$ to $\left(\frac{15}{4}, \frac{5\sqrt{3}}{4}\right)$ is $\frac{5\sqrt{3}}{2}$ using distance formula

Example 5: With Argument

$w = 4\sqrt{3} - 4i$. Find the minimum distance of w from the locus of $arg(z + 10i) = \frac{\pi}{3}$

Let's first draw a diagram.



$w = 4\sqrt{3} - 4i$

$arg(z + 10i) = \frac{\pi}{3}$

We want the distance from the complex number $4\sqrt{3} - 4i$ to the half line. The **minimum distance** is always the **perpendicular** distance hence right angle

Minimum distance is the length y shown

If we can find x we can find y since we can easily work out mag of $4\sqrt{3} - 4i$

Way 1: Geometric

We know the angle $\frac{\pi}{6}$ so we can build a triangle and use Pythagoras to find the length x

$$\sin\frac{\pi}{6} = \frac{x}{10}$$

$$x = 5$$

We know the magnitude of w

$$|w| = \sqrt{(4\sqrt{3})^2 + (-4)^2} = 8$$

$y = 8 - 5 = 3$

Min distance =3

Way 2: Algebraic

Way 1:	Way 2: Use template $y - b = \tan\theta(x - a)$
$arg(z + 10i) = \frac{\pi}{3}$ $arg(x + iy + 10i) = \frac{\pi}{3}$ $\tan\frac{\pi}{3} = \frac{y+10}{x}$ $\sqrt{3} = \frac{y+10}{x}$ $y = \sqrt{3}x - 10$	$y - -10 = \tan\frac{\pi}{3}(x - 0)$ $y + 10 = \sqrt{3}(x - 0)$ $y = \sqrt{3}x - 10$

The purple line is perpendicular since want the shortest distance

Note: w is just a point. We need to find the equation of the perpendicular line since we want the shortest distance and when we find the equation it happens to go through the origin

$$y = -\frac{1}{\sqrt{3}}x + c$$

Use the origin (0,0) or $(4\sqrt{3}, -4)$ to find c

$0 = -\frac{1}{\sqrt{3}}(0) + c$ hence $c = 0$

$$y = -\frac{1}{\sqrt{3}}x$$

Solving $y = \sqrt{3}x - 10$ and $y = -\frac{1}{\sqrt{3}}x$ simultaneously gives

$$\sqrt{3}x - 10 = -\frac{1}{\sqrt{3}}x$$

$$x = \frac{5}{2}\sqrt{3}$$

The intersection point is $\left(\frac{5}{2}\sqrt{3}, -\frac{5}{2}\right)$

Distance from $\left(\frac{5}{2}\sqrt{3}, -\frac{5}{2}\right)$ to $(4\sqrt{3}, -4)$ is 3 using distance formula

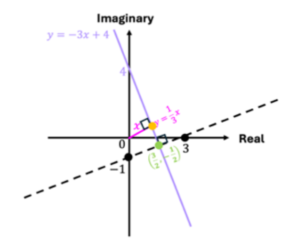
Example 6: With Perpendicular Bisector

$|z - 3| = |z + i|$. Find the least possible value of $|z|$

$|z - 3| = |z + i|$ is the perpendicular bisector of the line joining points $(3, 0)$ and $(0, -1)$

$|z|$ represents the distance from the origin

Minimum/shortest distance are always perpendicular distances so we want the pink distance below (labelled x) which is the **perpendicular distance** to the purple line from the ORIGIN.



The pink line is perpendicular to $y = -3x + 4$ which means it has gradient $\frac{1}{3}$ and passes through the origin

The equation is $y = \frac{1}{3}x$

We need the orange point which is the intersection therefore we solve $y = -3x + 4$ and $y = \frac{1}{3}x$ simultaneously

$$-3x + 4 = \frac{1}{3}x$$

$$x = \frac{6}{5}$$

Sub this back into either equation to get y $\Longrightarrow \left(\frac{6}{5}, \frac{2}{5}\right)$

Now we use distance formula to find the length

$$\text{Min distance} = \sqrt{\left(\frac{6}{5}\right)^2 + \left(\frac{2}{5}\right)^2} = 1.26$$

Min/Max Arguments

$arg(z)$ represents the angle the complex number makes with the real axis

Example 1: With A Circle

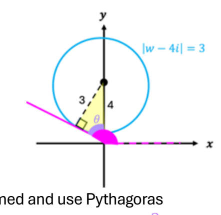
Given that $|w - 4i| = 3$

Find the maximum value of $arg\ z$ in the interval $(-\pi, \pi)$

$|w - 4i| = 3$ is a circle centre $(0,4)$ radius 3

$arg\ z$ represents the angle the complex number makes with the real axis. The angle found needs to be between $-\pi$ and π

Min/Max value of arg z occurs when the angle between the real x axis and the **tangent** to the circle is the smallest/largest (think of **sweeping until you hit the circle)**



Look at the triangle formed and use Pythagoras

$$\sin\theta = \frac{3}{4}$$

$$\theta = 0.848$$

$$\text{Max argument} = \frac{\pi}{2} + 0.848 = 2.42$$

Example 2: With A Circle

A complex number z is represented by the point P on the Argand diagram. Given that $|z - 5 - 3i| = 3$

i. Find the cartesian equation of the locus of P

ii. Find the maximum value of arg z in the interval $(-\pi, \pi)$

i.

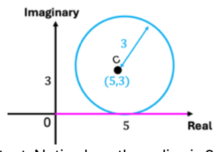
$|z - 5 - 3i| = 8$

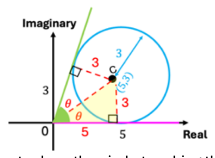
$|z - (5 + 3i)| = 3$

circle centre $(5, 3)$ radius 3

Cartesian equation is $(x-5)^2 + (y-3)^2 = 9$

ii.





Important: Notice how the radius is 3 so be sure to draw the circle touching the x axis

Max value of arg z occurs when the line between the origin and locus in a tangent to a circle . Note: if asked for min argument it is just 0 here.

As usual build a triangle with the origin and the centre of the circle

The line OC bisects the angle

Tangents are perpendicular to the radii of the circle hence we have the right angle and the 2 triangles formed are congruent

Use SOHCAHTOA on the yellow triangle

$$\tan\theta = \frac{3}{5} \text{ hence } \theta = \tan^{-1}\left(\frac{3}{5}\right) = 0.5404$$

$$\text{Max argument} = 2\theta = 2(0.5404) = 1.08$$

Example 3: With A Circle

Given that $|z + 2 - 2\sqrt{3}i| = 2$

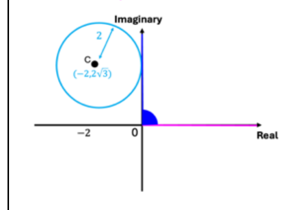
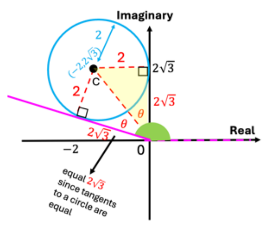
i. Write down the minimum value of arg z

ii. Find the maximum value of arg z

$|z - (-2 + 2\sqrt{3}i)| = 2$ is a circle centre $(-2, 2\sqrt{3})$ radius 2

$arg\ z$ represents the angle the complex number makes with the real axis. The angle found needs to be between $-\pi$ and π.

Max/Min value of arg z occurs when the angle between the real x axis and the tangent to the circle is the smallest/largest

Min arg z	Max arg z
Look for the smallest angle from the x axis to a tangent of the circle	Look for the largest angle from the x axis to a tangent of the circle
	
This is obvious and is just a right angle Min argument $= \frac{\pi}{2}$	Build a triangle with the origin and the centre of the circle Use SOHCAHTOA on yellow triangle $\tan\theta = \frac{2}{2\sqrt{3}}$ $\theta = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$ Max argument $= \frac{\pi}{2} + 2\theta = \frac{\pi}{2} + 2\left(\frac{\pi}{6}\right) = \frac{5\pi}{6}$

Example 4: With A Circle (Hard)

Let C be the subset of the complex plane defined by the equation $|z + 4 + 2i| = 2$. Sketch and find the point in C with the largest argument.

i.

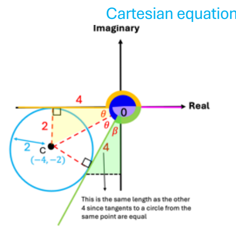
$|z + 4 + 2i| = 2$

$|z - (-4 - 2i)| = 2$

circle centre $(-4, -2)$ radius 2

Cartesian equation is $(x+4)^2 + (y+2)^2 = 4$

ii.



$arg\ z$ represents the angle the complex number makes with the real axis. The circle is in the second quadrant. The angle found needs to be between $-\pi$ and π.

Important: Notice how the radius is 2 so be sure to draw the circle touching the x axis

Use SOHCAHTOA on the yellow triangle

$$\tan\theta = \frac{2}{4} \text{ hence } \theta = \tan^{-1}\left(\frac{2}{4}\right) = 0.4636$$

We have 4 options:

Option 1: π

Option 2: $-\pi$

Option 3: $-(\pi - 2\theta) = -(\pi - 2(0.4636)) = -2.214$

Option 4: $\pi + 2\theta = \pi + 2(0.4636)$ =4.068 (this wins obviously since largest)

$$\cos\left(\frac{\pi}{2} - .4636 - .4636\right) = \frac{x}{4} \text{ and } \sin\left(\frac{\pi}{2} - .4636 - .4636\right) = \frac{y}{4}$$

$$z = x + iy = -2.4 - 3.2i$$

Note: You could have also solved $y = tan\ 4.068\ x$ simult with the circle

Example 5: (Very Hard)

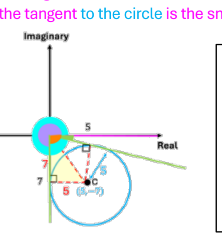
Given that $|z - 5 + 7i| = 5$. Find the maximum value of $arg\ z$ in the interval $(-\pi, \pi)$

i.

$|z - (5 - 7i)| = 5$ is a circle centre $(5, -7)$ radius 5

$arg\ z$ represents the angle the complex number makes with the real axis. The angle found needs to be between $-\pi$ and π.

Max/Min value of arg z occurs when the angle between the real x axis and the tangent to the circle is the smallest/largest



Important: Notice how the radius is 5 so be sure to draw the circle touching the y axis

The argument can be all possible options shown above (purple, turquoise, light green or orange. Question says we need it to be $(-\pi, \pi)$ hence the greatest is the green angle

Use SOHCAHTOA on the yellow triangle

$$\tan\theta = \frac{5}{7} \Rightarrow \theta = \tan^{-1}\left(\frac{5}{7}\right) = 0.6202$$

This time we have to go clockwise since we can't have an angle greater than π and going anticlockwise would produce this

$$\text{Max argument} = -\left(\frac{\pi}{2} - 2\theta\right) = -\left(\frac{\pi}{2} - 2(0.6202)\right) = -0.330$$

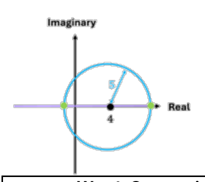
Intersections

Example 1:

Given that z satisfies $|z - 4| = 5$

Find the values of z that satisfy both $|z - 4| = 5$ and $Im(z) = 0$

$|z - 4| = 5$ and $Im(z) = 0$



$Im(x + iy) = 0$

$y = 0$

We need to find the green points

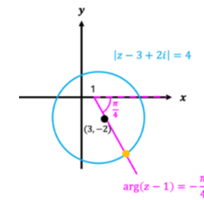
Way 1: Geometrical	Way 2: Algebraic
$(4,0)$ plus or minus length 5 in both directions Points are $(9,0)$ and $(-1,0)$	$(x-4)^2 + y^2 = 25$ $y = 0$ Solve simultaneously $(x-4)^2 + 0^2 = 25$ $(x-4)^2 = 25$ $x - 4 = \pm 5$ $x = 5 + 4, x = 5 - 4$ $x = 9, x = -1$ Points are $(9,0)$ and $(-1,0)$ $z = 9, z = -1$

Example 2:

$|z - 3 + 2i| = 4$ and $\arg(z - 1) = -\frac{\pi}{4}$

Given that $z = a + ib$, where $a, b \in \mathbb{R}$

Find the exact value of a and the exact value of b



Way 1: Geometrical	Way 2: Algebraic
$(x-3)^2 + (y+2)^2 = 16$	We need the equation of the line for the argument $\arg(z-1) = -\frac{\pi}{4}$ $\arg(x + iy - 1) = -\frac{\pi}{4}$ $\arg((x-1) + iy) = -\frac{\pi}{4}$ This is the half line in quadrant 4 $\tan\left(-\frac{\pi}{4}\right) = \frac{y}{x-1}$ $-1 = \frac{y}{x-1}$ $y = -x + 1$

We need to find the orange point hence solve simultaneously

$(x-3)^2 + (y+2)^2 = 16$ and $y = -x + 1$

$x = 3 \pm 2\sqrt{2}, y = -2 \pm 2\sqrt{2}$

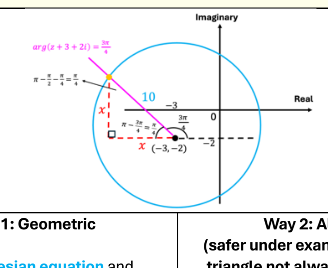
But $x > 1, y < 0$ from diagram

$x = 3 + 2\sqrt{2}$ and $y = -2 - 2\sqrt{2}$ which gives $z = 3 + 2\sqrt{2}, +(-2 - 2\sqrt{2})i$

$a = 3 + 2\sqrt{2}, b = -2\sqrt{2} - 2$

Example 3:

Find the complex number z that satisfies both $|z + 3 + 2i| = 10$ and $arg\ (z + 3 + 2i) = \frac{3\pi}{4}$



Way 1: Geometric	Way 2: Algebraic (safer under exam pressure since triangle not always easy to spot)
Draw the cartesian equation and then we can form a triangle Triangle is isosceles since both angles are $\frac{\pi}{4}$ $x^2 + x^2 = 10^2$ $2x^2 = 100$ $x^2 = 50$ $x = \sqrt{50}$ $x = 5\sqrt{2}$ To find the orange point we can go backwards rom the centre $(-3 - 5\sqrt{2}, -2 + 5\sqrt{2})$ $-3 - 5\sqrt{2} + i(-2 + 5\sqrt{2})$	Solve $y = -x - 5$ $(x+3)^2 + (y+2)^2 = 100$ Solve simultaneously $(x+3)^2 + (-x - 5 + 2)^2 = 100$ $(x+3)^2 + (-x-3)^2 = 100$ $2x^2 + 12x - 82 = 0$ $x^2 + 6x - 41 = 0$ $x = \frac{-6 \pm \sqrt{36 - 4(1)(-41)}}{2(1)}$ $x = -3 \pm 5\sqrt{2}$ Sub back into $y = -x - 5$ $y = -(-3 \pm 5\sqrt{2}) - 5$ $y = -2 \mp 5\sqrt{2}$ Hence we have the points $(-3 \pm 5\sqrt{2}, -2 \mp 5\sqrt{2})$ $z = -3 + 5\sqrt{2} + i(-2 - 5\sqrt{2})$ $z = -3 - 5\sqrt{2} + i(-2 + 5\sqrt{2})$ We want the second quadrant solution so neg real and pos imaginary, hence $z = -3 - 5\sqrt{2} + i(-2 + 5\sqrt{2})$

Example 4: Hard

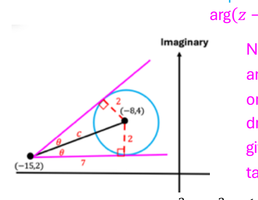
A complex number z is represented by the point P on the argand diagram. Given $|z + 8 - 4i| = 2$

i. Show that the maximum value of $\arg(z + 15 - 2i)$ in the interval $(-\pi, \pi)$ is $2arcsin\left(\frac{2}{\sqrt{53}}\right)$

ii. Find the exact values of the complex numbers that satisfy both $|z + 8 - 4i| = 2$ and $\arg(z + 4i) = \frac{3\pi}{4}$

$|z - (-8 + 4i)| = 2$

$\arg(z - (-15 + 2i)) = \theta$



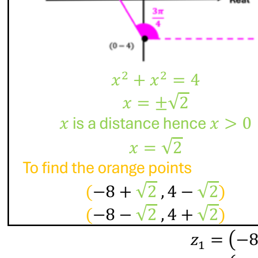
Not a max arg z question now. Instead, a max arg $(z - (-15 + 2i)$ hence the line won't be on x axis. We know the radius is 2 and dropping down from the y coord of the centre gives y coord of 2 hence the line y=2 is a tangent to the circle

i. $c^2 = 4^2 + (7)^2$ hence $c = \sqrt{53}$

$$\sin\theta = \frac{2}{\sqrt{53}} \text{ hence } \theta = \sin^{-1}\left(\frac{2}{\sqrt{53}}\right)$$

$$2\theta = 2\sin^{-1}\left(\frac{2}{\sqrt{53}}\right)$$

ii.

Way 1: Geometric	Way 2: Algebraic
$\lvert z + 8 - 4i\rvert = 2$ $\arg(z - (-4i)) = \frac{3\pi}{4}$  $x^2 + x^2 = 4$ $x = \pm\sqrt{2}$ x is a distance hence $x > 0$ $x = \sqrt{2}$ To find the orange points $(-8 + \sqrt{2}, 4 - \sqrt{2})$ $(-8 - \sqrt{2}, 4 + \sqrt{2})$	$\arg(z + 4i) = \frac{3\pi}{4}$ $\arg(x + iy + 4i) = \frac{3\pi}{4}$ $\arg(x + i(y + 4)) = \frac{3\pi}{4}$ $\tan\frac{3\pi}{4} = \frac{y+4}{x}$ $-1 = \frac{y+4}{x}$ $x = -y - 4$ $(x+8)^2 + (y-4)^2 = 4$ Solve simultaneously $(-y - 4 + 8)^2 + (y-4)^2 = 4$ $(-y+4)^2 + (y-4)^2 = 4$ $2y^2 - 16y + 28 = 0$ $y = 4 \pm \sqrt{2}$ $x = 4 \mp \sqrt{2}$

$$z_1 = (-8 + \sqrt{2}) + (4 - \sqrt{2})i$$

$$z_2 = (-8 - \sqrt{2}) + (4 + \sqrt{2})i$$

Number of Solutions

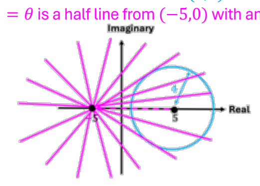
(question can ask for range of values where two solutions, one solution or no solution)

Example 1:

Given that $\arg(z + 5) = \theta$ and $|z - 5| = 4$ have no common solutions, find **range of possible values** of θ in interval $(-\pi, \pi)$

$|z - 5| = 4$ which is a circle centre $(5,0)$ radius 4

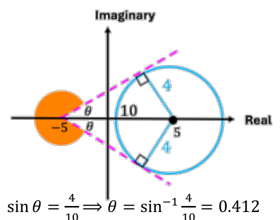
$\arg(z + 5) = \theta$ is a half line from $(-5,0)$ with angle θ



We can see from the picture above that there are many options of the angle coming off of the dashed half line (any anticlockwise angle). We don't know the angle θ. We also know the angle has to be between $-\pi$ and π

But we do not want a common solution hence we **do not** want the line to intersect the circle.

We can have any angle value within the following dashed lines. Let's find θ



$$\sin\theta = \frac{4}{10} \Rightarrow \theta = \sin^{-1}\frac{4}{10} = 0.412$$

θ needs to be beyond the angle 0.412 but less than the angle $2\pi - 0.412 = 5.871$

Hence

$$0.412 < \theta < 5.871$$

We need interval $(-\pi, \pi)$ hence $0.412 < \theta < \pi$ or $-\pi < \theta < -0.412$

Notice how there is no equality on the inequality since we do not want to touch the circle

Note: We can have 3 types of questions which mention number of solutions. We draw 2 tangents to the circle

- **At** the tangents there will be **1 solution** $\theta = a$, $\theta = b$
- **In between** the tangents there will be **2 solutions** $\theta < a$ and $\theta > b$
- **Beyond** the tangents there will be **no solutions** $a < \theta < b$

So, we find the value of θ **at** the tangents and then we can state the range of values of θ needed to be in the correct case above

Example 2:

Given that the complex number z satisifes the equation $|z + 6 + 6i| = 4$. Find the range of values for $\theta, -\pi < \theta < \pi$, for which $\arg(z - 4 + 2i) = \theta$ and $|z + 6 + 6i| = 4$ have no solutions

$\arg(z - 4 + 2i) = \theta$

$\arg(z - (4 - 2i)) = \theta$

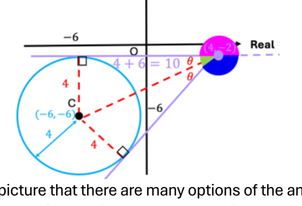
half line from $(4, -2)$ with angle θ

$|z + 6 + 6i| = 4$

Circle centre $(-6, -6)$ radius 4

Hint for below:

We need a horizontal length like in the last question and we can use the fact that the radius of the circle being 4 will bring us up to the same y coordinate as the half line



We can see from the picture that there are many options of the angle coming off of the dashed half line (any anticlockwise angle). We don't know the angle θ

But we do not want a common solution hence we **do not** want the line to intersect the circle.

$$\tan\theta = \frac{4}{10}$$

$$\theta = tan^{-1}\frac{4}{10} = 0.3805$$

$$2\theta = 0.761$$

Half-line **does not** intersect the circle when

- $-\pi + 0.761 \le 0$ which is the same as $-2.38 \le \theta \le 0$
- $0 < \theta < \pi$

Hence no solutions when $-2.38 < \theta < 0$ and $0 < \theta < \pi$

(only have solutions when $-\pi \le \theta \le 2.38$)

Note 1: We could have also given the answer as $-2.38 < \theta < \pi$, but since we want the argument to not be more than π or less than π the first solution is best

Note 2: You could have also found the cartesian equations and used the fact that the discriminant is less than 0

Example 3: Very Hard (October 2020 AS Core Q10)

There are two distinct complex numbers z that satisfy

$$\{z: |z - 3 - 5i| = 2r\} \cap \left\{z: \arg(z - 2) = \frac{3\pi}{4}\right\}$$

Determine the exact range of values for the real constant r

$|z - 3 - 5i| = 2r$

This is a circle with equation $(x-3)^2 + (y-5)^2 = 4r^2$

$$\arg(z - 2) = \frac{3\pi}{4}$$

$$\arg(x + iy - 2) = \frac{3\pi}{4}$$

$$\tan\frac{3\pi}{4} = \frac{y}{x-2}$$

$$y = -x + 2$$

Solve simultaneously

$$(x-3)^2 + (y-5)^2 = 4r^2$$

$$(x-3)^2 + (-x + 2 - 5)^2 = 4r^2$$

$$2x^2 + 18 - 4r^2 = 0$$

2 distinct roots therefore discriminant $b^2 - 4ac > 0$

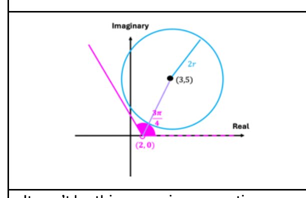
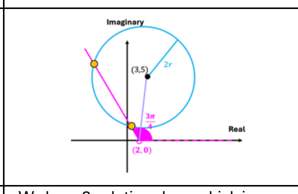
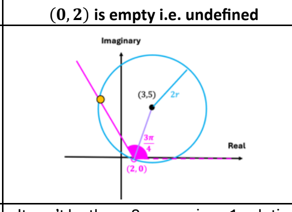
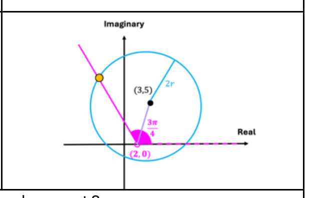
$$0^2 - 4(2)(18 - 4r^2) > 0$$

$$-144 + 32r^2 > 0$$

$$r < -\frac{3\sqrt{2}}{2} \text{ or } r > \frac{3\sqrt{2}}{2}$$

Radius can't be negative so $r > \frac{3\sqrt{2}}{2}$

This ensures 2 solutions for an intersection with the line and circle, but the line is only a **half line** and stops (it is not a continuous line like usual which the simultaneous equation assumes), so we need to think of the graph. The options we can have as follows:

Radius of circle less than purple length		Radius of circle same as purple length	Radius of circle bigger than purple length
Case 1: No solution	**Case 2: 2 solutions**	**Case 3: 1 solution** $(0, 2)$ **is empty i.e. undefined**	**Case 4: 1 solution**
			
It can't be this case since question says need 2 distinct solutions	We have 2 solutions here which is what we want hence we want	It can't be these 2 cases since 1 solution and we want 2	

Recall the circle has radius $2r$. In order to have 2 solutions we also need the radius to be less than what the distance would be from (3,5) to (2,0). This picks up case 1, 2 and 3 but case 1 and 3 has been disregarded already with the discriminant

$$\text{Distance from centre to (2,0)} = \sqrt{(3-2)^2 + (5-0)^2} = \sqrt{26}$$

Hence, we need $2r < \sqrt{26}$ which gives $r < \frac{\sqrt{26}}{2}$

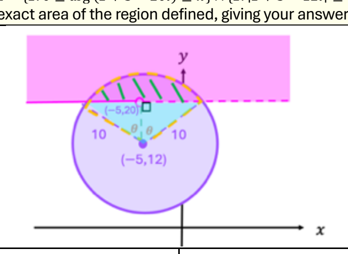
$$\frac{3\sqrt{2}}{2} < r < \frac{\sqrt{26}}{2}$$

Areas Of Regions

Example 1: (June 2024 AS Core Q5)

$A = \{z: 0 \le \arg(z + 5 - 20i) \le \pi\} \cap \{z: |z + 5 - 12i| \le 10$

Determine the exact area of the region defined, giving your answer in simplest form



Area of isosceles triangle:	Area of sector:
Subtract the y-coordinate of the centre of the circle from the starting point of the half-line to obtain the height of the triangle. $height = 20 - 12 = 8$ We can split the triangle in half and use Pythagoras to calculate half the length of the base of the isosceles triangle. $half\ the\ total\ base = \sqrt{10^2 - 8^2} = 6$ $total\ length\ of\ base = 6 \times 2 = 12$ Therefore, the area of the isosceles triangle is: $\frac{1}{2}bh = \frac{1}{2}(12)(8) = 48$	We can use the lengths obtained in calculating the area of isosceles triangle and use trig to obtain the value for θ $\sin\theta = \frac{6}{10}$ or $\cos\theta = \frac{8}{10}$ $\theta = 0.6435\ rad$ Use the formula for the area of a sector $\frac{1}{2}r^2\theta$ $= \frac{1}{2}(10)^2(0.6435..)$ $= 32.175 ...$ hence $2\theta = 64.35$

We want area of the green region

= area of the sector − area of the isosceles triangle

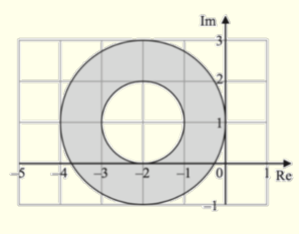
Therefore, the total area is

$$64.35 - 48 = 16.4\ (3sf)$$

$1 \le |z + 2 - i| \le 2$ $\quad |z - i| \le |z - 3i|$

Example 2: Very Hard (October 2021 AS Core Q5)

The diagram below shows an argand diagram.



The set P, of point that lie within the shaded region including its boundaries, is defined by

$$P = \{z \in \mathbb{C}: a \le |z + b + ci| \le d\}$$

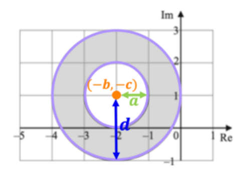
i. Write down the values of a, b, c and d where a, b, c and d are integers

The set Q is defined by

$$Q = \{z \in \mathbb{C}: a \le |z + b + ci| \le d\} \cap \{z \in \mathbb{C}: |z - i| \le |z - 3i|\}$$

ii. Determine the exact area of the region defined by Q, giving your answer in simplest form

i.



$$a \le |z + b + ci| \le d$$

$$a \le |z - (-b - ci)| \le d$$

We can clearly see from the diagram

$(-b, -c) = (-2, 1)$

$b = 2$

$c = -1$

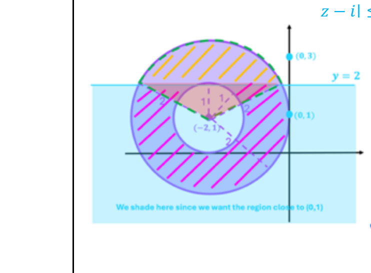
$a = 1$

$d = 2$

ii.

$1 \le |z - (-2 + i)| \le 2$

$z - i| \le |z - 3i|$

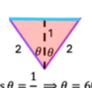


Purple area $= \pi(2)^2 - \pi(1)^2 = 3\pi$

Pink area (what we want)

Orange area is a segment which we always get by doing **sector − triangle**

Sector $= \frac{\theta}{360} \times \pi(2)^2$



$\cos\theta = \frac{1}{2} \Rightarrow \theta = 60$

Area of triangle $= \frac{1}{2}(2)(2)\sin 120 = \sqrt{3}$

Segment area $= \frac{120}{360} \times \pi(2)^2 - \sqrt{3}$

pink area $= 3\pi - \left(\frac{120}{360} \times \pi(2)^2 - \sqrt{3}\right)$

$$= 3\pi - \frac{4}{3}\pi + \sqrt{3} = \frac{5}{3}\pi + \sqrt{3}$$